

DUAL SPACES OF WEIGHTED SPACES

BY
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Abstract. The topological duals of a large class of weighted spaces of continuous functions are characterized as spaces of Radon measures which can be factored into a product of a weight function and a bounded Radon measure. We next obtain a representation for a base for the equicontinuous subsets of these dual spaces and for the extremal points of the members of this base. Finally, among other applications, these ideas make possible an extension of the representation theorem for biequicontinuous completed tensor products of weighted spaces obtained by the author in an earlier paper.

1. Introduction. The role of weighted spaces of continuous functions in approximation theory is well known (e.g., see Nachbin [7], [8]). Moreover, weighted spaces as locally convex spaces have proved to be a useful tool in the study of entire functions (see Taylor [14]) and other mathematical objects. For example, Buck [1], Collins and Dorroh [2], and Conway [3], [4], among others, have implemented the notion that the weighted space $(C_b(X), \beta)$ of bounded continuous functions on a locally compact Hausdorff space X with the strict topology β is the proper setting to study the space $C_b(X)$ as a locally convex space if one wishes to relate its properties with those of X . One of the key results in this direction was the identification by Buck [1] of the topological dual $(C_b(X), \beta)^*$ with the space of bounded Radon measures on X . We should also mention that a recent representation theorem due to Prolla [9] manifests the universal nature of weighted spaces in the class of all locally convex spaces.

It is the purpose of this paper to continue the investigation of weighted spaces as locally convex spaces that was initiated in [13]. In particular, it is shown that our characterization of the topological duals of certain weighted spaces as spaces of Radon measures in [13] can be extended to a larger class of weighted spaces (we are now able to remove a continuity condition imposed in [13]), and several examples are given to help delineate the situation. Moreover, we give characterizations analogous to those given in [13] for a base for the equicontinuous sets in the topological duals of this larger class of weighted spaces and the extremal points of the members of this base. Finally, various applications of the preceding results

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are made; these include a generalization of our representation theorem for the biequicontinuous completed tensor product of weighted spaces given in [13].

The contents of this paper are, for the most part, either results or generalizations of results from the author's Ph.D. dissertation written under the direction of Professor Heron S. Collins at Louisiana State University (1968). The author wishes to express his gratitude to Professor Collins for his advice and encouragement.

2. Preliminaries. The underlying topological spaces (X and Y) in the sequel will be at least completely regular T_1 -spaces, while all functions will be complex-valued unless explicitly stated otherwise. We shall let $C(X)$ denote the space of all continuous functions on X , $B(X)$ denote the space of all bounded functions on X , and $B_0(X)$ denote the space of all functions on X which vanish at infinity. Further, let $C_b(X) = C(X) \cap B(X)$ and $C_0(X) = C(X) \cap B_0(X)$, while $C_c(X)$ will denote the space of all continuous functions on X which have compact support. If f is a function on X , then $N(f) = \{x \in X : f(x) \neq 0\}$ and the closure of $N(f)$ is denoted $\text{spt}(f)$. Moreover, if $S \subseteq X$, then $R(f; S)$ will denote the restriction of f to S , and if $R(f; S) \in B(S)$, then define $\|f\|_S = \sup \{|f(x)| : x \in S\}$. In the case $S = X$ we write $\|f\|$, and will refer to the topology induced by $\|\cdot\|$ on $B(X)$ as the uniform topology.

A *Nachbin family* V on X is a set of nonnegative upper semicontinuous (u.s.c.) functions on X satisfying the condition that if $u, v \in V$ and $\lambda \geq 0$, then there is a $w \in V$ such that $\lambda u, \lambda v \leq w$ (pointwise). If V is a Nachbin family on X , then the corresponding *weighted space*

$$CV_0(X) = \{f \in C(X) : fv \in B_0(X) \text{ for every } v \in V\}$$

endowed with the *weighted topology* ω (or ω_V) generated by the seminorms $\{p_v : v \in V\}$ where $p_v(f) = \|fv\|$ for each $f \in CV_0(X)$. Now $CV_0(X)$ is a locally convex (topological vector) space with a base of closed and absolutely convex neighborhoods of zero formed by the sets $V_v = \{f \in CV_0(X) : \|fv\| \leq 1\}$ for each $v \in V$. If U and V are two Nachbin families on X and if for each $u \in U$ there is a $v \in V$ such that $u \leq v$, then we write $U \leq V$. In case $U \leq V$ and $V \leq U$, we write $U \approx V$.

Unless otherwise specified, we will consider only Nachbin families V on X for which $CV_0(X)$ is Hausdorff (necessary and sufficient conditions for this, as well as examples of weighted spaces are given in [13]). We should note at this point that requiring $CV_0(X)$ to be Hausdorff can yield information about X in certain cases. In fact, requiring $CV_0(X)$ to be Hausdorff can be very close to requiring that X be locally compact.

2.1. THEOREM. *If V is a Nachbin family on X with $C_c^+(X) \leq V \leq C_0^+(X)$, then $CV_0(X)$ is Hausdorff if and only if X contains a dense locally compact (and hence open) subspace Y .*

Proof. Assume Y is a dense locally compact subspace of X ; thus Y is open in X [6, p. 45]. If $f \in CV_0(X)$ with $f \neq 0$, then there is an $x \in Y$ with $f(x) \neq 0$, and hence there is an open neighborhood A of x with $A \subseteq Y$ and such that $\text{cl}(A)$ is compact. Consequently, there exists a $\phi \in C_c^+(X)$ with $\phi(x) > 0$, and this implies there is a $v \in V$ such that $\phi \leq v$ from which it follows that $|f(x)|v(x) > 0$.

Now assume $CV_0(X)$ is Hausdorff. If $y \in X$ and if A is an open neighborhood of y , then there exists a $\phi \in C(X)$ such that $0 \leq \phi \leq 1$, $\phi(y) = 1$, and $\phi(X \setminus A) = 0$. If $v \in V$ and $\varepsilon > 0$, then $K = \{x \in X : \phi(x)v(x) \geq \varepsilon\} \subseteq \{x \in X : v(x) \geq \varepsilon\}$, and this implies K is compact. Hence $\phi \in CV_0(X)$, and since $\phi \neq 0$ there is a $v \in V$ with $\phi v \neq 0$; i.e., $N(v) \cap A$ is not void. Therefore $Y = \bigcup \{N(v) : v \in V\}$ is dense in X ; and if $y \in Y$, then there exists $v \in V$ such that $y \in B = \{x \in X : v(x) > (1/2)v(y)\}$. But B is open, $\text{cl}(B)$ is compact, and $\text{cl}(B) \subseteq Y$ which completes the proof.

For example, if we let X denote the rationals with the topology induced by the usual topology on the reals, then X is a completely regular T_1 -space which does not have a dense locally compact subspace. In particular, if V is a Nachbin family on X with $C_c^+(X) \leq V \subseteq C_0^+(X)$, then $V = \{0\}$ and $CV_0(X)$ is $C(X)$ with the indiscrete topology.

3. The dual space of $CV_0(X)$. In [13], the author characterized the space $CV_0(X)^*$ in the case V was a Nachbin family on a locally compact space X which satisfied the condition that $V \subseteq C^+(X)$. We are now able to generalize this result to a much larger class of Nachbin families on locally compact spaces (Theorem 3.1).

We will denote the complex Radon measures on a locally compact space X by $M(X)$, while $M_b(X)$ will denote the subspace of bounded Radon measures. Now if V is a Nachbin family on X , then each $v \in V$ is Borel measurable and bounded on the compact subsets of X . Define $V \cdot M_b(X)$ to be $\{v \cdot \mu : v \in V, \mu \in M_b(X)\}$ where $v \cdot \mu$ is the Radon measure on X defined by $v \cdot \mu(f) = \mu(fv)$ for $f \in C_c(X)$. Therefore $V \cdot M_b(X)$ is always contained in $M(X)$, and will actually be a linear subspace of $M(X)$ for a large class of Nachbin families V on X , although this is by no means obvious from the definition. In fact, $V \cdot M_b(X)$ is our candidate for $CV_0(X)^*$ in the case of the Nachbin families to be considered, but, of course, we do not expect uniqueness of representation (see Example 3.6).

3.1. THEOREM. *If X is locally compact and if V is a Nachbin family on X with $V \subseteq C^+(X)$, then $V \cdot M_b(X)$ is a linear subspace of $M(X)$ and $T: V \cdot M_b(X) \rightarrow CV_0(X)^*$ is a (linear) onto isomorphism where $T(v \cdot \mu)(f) = \int fv d\mu$ for each $f \in CV_0(X)$. In this case we write $CV_0(X)^* = V \cdot M_b(X)$.*

The proof of Theorem 3.1 will be deferred until after the following lemmas have been established.

3.2. LEMMA. *If X is locally compact and if $V = C^+(X)$, then $C_c(X) \subseteq CV_0(X) \subseteq C_b(X)$, ω_V is finer than the uniform topology on $CV_0(X)$, and the inductive limit topology on $C_c(X)$ is finer than the topology induced on $C_c(X)$ by ω_V .*

Proof. $C_c(X)$ is always contained in $CV_0(X)$, while it is clear in this case that $CV_0(X) \subseteq C_b(X)$. Moreover, it is immediate from [13, Theorem 3.1] that ω_V is finer than the uniform topology. Now let $v \in V$, and for every compact K in X let $B_K = \{f \in C_c(X) : \text{spt}(f) \subseteq K, \|f\| \leq (\|v\|_K)^{-1}\}$ if $N(v) \cap K$ is nonvoid and let $B_K = \{f \in C_c(X) : \text{spt}(f) \subseteq K\}$ if $N(v) \cap K$ is void. Since the absolutely convex hull A of $\bigcup \{B_K : K \text{ compact}\}$ is an inductive limit neighborhood of zero in $C_c(X)$ [10, p. 79], it will suffice to show that $f \in A$ implies $f \in V_v \cap C_c(X)$. If $f \in A$, then there exist scalars $\{\lambda_k\}_{k=1}^n$ and functions $\{f_k\}_{k=1}^n$ with $\sum_{k=1}^n |\lambda_k| \leq 1$, and $f_k \in B_{K_k}$, $k=1, \dots, n$, such that $f = \sum_{k=1}^n \lambda_k f_k$. Hence

$$|f(x)|v(x) \leq \sum_{k=1}^n |\lambda_k| \cdot |f_k(x)|v(x) \leq \sum_{k=1}^n |\lambda_k| \leq 1,$$

and this implies $\|fv\| \leq 1$.

3.3. LEMMA. *Let X be locally compact and let v be a nonnegative u.s.c. function on X . If $v \in M^+(X)$ has the property that $\int f dv \leq 1$ for every $f \in C_c^+(X)$ with $\|fv\| \leq 1$, then there exists $\mu \in M_b^+(X)$ with $\|\mu\| \leq 1$ and such that $v = v \cdot \mu$.*

Proof. Define u on X by

$$\begin{aligned} u(x) &= (1/v)(x), & x \in N(v), \\ &= +\infty, & x \in X \setminus N(v), \end{aligned}$$

and note that u is a nonnegative lower semicontinuous function on X . If $\phi \in C_c^+(X)$ and if $\phi \leq u$, then $\|\phi v\| \leq 1$ which implies $\int \phi dv \leq 1$. Consequently,

$$\sup \left\{ \int \phi dv : \phi \in C_c^+(X), \phi \leq u \right\} \leq 1,$$

and this implies the upper integral of u with respect to v is finite. Hence u is v -integrable from which it follows that $v(X \setminus N(v)) = 0$ and $\int u dv = \int (1/v) dv \leq 1$; i.e., the measure $u \cdot v = (1/v) \cdot v$ is the desired μ .

Proof of 3.1. Since $C_c(X)$ is an ω_V -dense subspace of $CV_0(X)$ [8, p. 64], it follows that T is a well-defined one-to-one function into $CV_0(X)^*$. We next show that T is onto, and to this end, fix $L \in CV_0(X)^*$. Now the inductive limit topology on $C_c(X)$ is finer than the topology induced on $C_c(X)$ by ω_V from Lemma 3.2 and [13, Theorem 3.1], and so $R(L; C_c(X)) \in (C_c(X), \text{ind lim})^*$. Therefore there exists a unique $\mu \in M(X)$ such that $R(L; C_c(X))(f) = \int f d\mu$ for $f \in C_c(X)$. Write μ uniquely as $\alpha + i\beta$ where α and β are real Radon measures on X , and let $\alpha = \alpha^+ - \alpha^-$ be the minimal decomposition of α . Since $L \in CV_0(X)^*$, there is a $v \in V$ such that $|L(f)| \leq 1$ for every $f \in V_v$; thus if $f \in C_c(X) \cap V_v$, then $|\int f d\alpha| \leq |\int f d\mu| \leq 1$ from whence $\int f d\alpha^+ \leq 1$. By Lemma 3.3 there is a $\sigma \in M_b(X)$ such that $\alpha^+ = v \cdot \sigma$ (with $\|\sigma\| \leq 1$), and it easily follows that there is a $\tau \in M_b(X)$ such that $\alpha^- = v \cdot \tau$. It is now clear that there exists a $\nu \in M_b(X)$ such that $\mu = v \cdot \nu$ (we will show in §4 that ν can be chosen so that $\|\nu\| \leq 1$). Hence $\mu \in V \cdot M_b(X)$, and since $T(\mu)$ agrees with L

on an ω_V -dense subspace of $CV_0(X)$ we have that $T(\mu)=L$. It remains to show $V \cdot M_b(X)$ is a linear subspace of $M(X)$ and that T is linear. To see this, first note that $V \cdot M_b(X)$ is closed under scalar multiplication. Now if $\mu, \nu \in V \cdot M_b(X)$, then $T(\mu)+T(\nu) \in CV_0(X)^*$, and hence there exists a $\tau \in V \cdot M_b(X)$ such that $T(\tau) = T(\mu)+T(\nu)$. Taking $f \in C_c(X)$, we have

$$\tau(f) = T(\tau)(f) = (T(\mu)+T(\nu))(f) = (\mu+\nu)(f)$$

which implies $\tau = \mu + \nu$ and concludes the proof.

We now proceed to show that Theorem 3.1 does indeed generalize [13, Theorem 3.9]. To do this we utilize a topological space considered by Shiota in [12].

3.4. EXAMPLE. Denote the first uncountable ordinal by Ω , the first infinite ordinal by ω , and the space of all ordinals less than an ordinal α with the order topology by $W(\alpha)$. Define

$$X = W(\Omega+1) \times W(\omega+1) \setminus A$$

where $A = \{(\alpha, \omega) : \alpha \text{ is a limit ordinal in } W(\Omega+1)\}$, and note that X is locally compact and Hausdorff. Now let \mathcal{P} denote the collection of all closed and relatively precompact (for a definition, see [12]) subsets of X , and for each $P \in \mathcal{P}$ let χ_P be the characteristic function of P . If $V = \{\lambda \chi_P : \lambda \geq 0, P \in \mathcal{P}\}$, then V is a Nachbin family on X with $V \leq C^+(X)$. Moreover, if there were a Nachbin family U on X with $U \leq C^+(X)$ and $U \approx V$, then $f \in U$ would imply that $\text{spt}(f)$ is relatively precompact. But $P = \{(\Omega, \alpha) : \alpha \in W(\omega)\}$ is in \mathcal{P} while if $f \in C^+(X)$ with $f(x) \geq 1$ for $x \in P$, then $\text{spt}(f)$ is not relatively precompact.

It is the case, of course, that some restrictions are imposed on the class of Nachbin families under consideration in Theorem 3.1. One of these is the assumption that $CV_0(X)$ be Hausdorff which introduces in some sense a lower bound on this class, but will not exclude, for example, any V for which ω_V is at least as fine as pointwise convergence. The other restriction is the hypothesis $V \leq C^+(X)$, and the following example suggested by Professor N. Kimura shows this condition is not redundant.

3.5. EXAMPLE. If we let

$$X = W(\Omega+1) \times W(\omega+1) \setminus \{(\Omega, \omega)\} \quad (\text{the Tychonoff plank})$$

and define v on X by

$$\begin{aligned} v(\alpha, \beta) &= 0, & \alpha &\neq \Omega, \\ &= \beta, & \alpha &= \Omega, \end{aligned}$$

then v is a nonnegative u.s.c. function on X which is unbounded, while $f \in C^+(X)$ implies f is bounded.

We will conclude this section with a remark concerning $CV_0(X)^*$ where V and X are as in Theorem 3.1. As we have seen, in this case $CV_0(X)^* \leq M(X)$ and $\mu \in CV_0(X)^*$ implies there exist $v \in V$, $\nu \in M_b(X)$ so that $\mu = v \cdot \nu$ (a factorization

of μ !). In general, this factorization is not unique as we show in the following example.

3.6. EXAMPLE. Let X denote the reals with the usual topology, let m denote Lebesgue measure on X , let $u(x) = \exp(|x|)$, and let $v(x) = \exp(2|x|)$. Now $u, v, 1/u, 1/v \in C^+(X)$ and $\mu = (1/u) \cdot m, \nu = (1/v) \cdot m \in M_b(X)$ with $\mu \neq \nu$, but $m = u \cdot \mu = v \cdot \nu$.

4. **Equicontinuous sets in $CV_0(X)^*$.** In this section we will obtain a useful representation of a base for the equicontinuous subsets of $CV_0(X)^*$, namely $\{V_v^0 : v \in V\}$; and then characterize the extremal points $\mathcal{E}(V_v^0)$ of the members of this base. It is not difficult to see that if V is a Nachbin family on a locally compact space X with $V \subseteq C^+(X)$ and if $B = \{\mu \in M_b(X) : \|\mu\| \leq 1\}$, then V_v^0 is the weak-* closure in $CV_0(X)^*$ of $v \cdot B$. We will show that $v \cdot B$ is weak-* closed whence $V_v^0 = v \cdot B$.

The following (probably known) lemma is included because of its role in the sequel.

4.1. LEMMA. *If X is locally compact, if u is a nonnegative u.s.c. function on X , and if there is an $f \in C^+(X)$ such that $u \leq f$, then $u = \inf \{f \in C^+(X) : f \geq u\}$.*

Proof. Fix $y \in X$, let $\varepsilon > 0$, and choose $f \in C^+(X)$ with $f \geq u$. We may assume $f(y) - u(y) \geq \varepsilon$. Now $A = \{x \in X : u(x) < u(y) + \varepsilon/4\}$ and $B = \{x \in X : f(x) > f(y) - \varepsilon/4\}$ are both open and $y \in A \cap B$. Take $\phi \in C_c(X)$ with $\phi(y) = 1$, $0 \leq \phi \leq 1$, and $\text{spt}(\phi) \subseteq A \cap B$; define $\theta = (f(y) - u(y) - 3\varepsilon/4)\phi$ and $g = f - \theta$. Thus $g \in C(X)$ and $g(y) - u(y) = 3\varepsilon/4 < \varepsilon$. Moreover, $x \in X \setminus (A \cap B)$ implies $g(x) - u(x) = f(x) - u(x) \geq 0$, while $x \in A \cap B$ implies $g(x) = f(x) - \theta(x) > f(y) - \varepsilon/4 - (f(y) - u(y) - 3\varepsilon/4)\phi(x) \geq u(y) + \varepsilon/2 > u(x)$.

4.2. LEMMA. *If X is locally compact, if $f, g \in C^+(X)$, and if for some $y \in X$ we have $f(y)g(y) < 1$, then there exists $\phi \in C_c^+(X)$ satisfying $\phi(y) = g(y)$, $\phi \leq g$, and $\|f\phi\| \leq 1$.*

Proof. If $A = \{x \in X : f(x)g(x) < 1\}$, then there exists $\theta \in C_c(X)$ so that $0 \leq \theta \leq 1$, $\theta(y) = 1$, and $\text{spt}(\theta) \subseteq A$. It now follows that $\phi = g\theta$ is the desired function.

4.3. LEMMA. *Let X be locally compact, let u be a nonnegative u.s.c. function on X for which there is an $f \in C(X)$ such that $u \leq f$, let $y \in X$, and let $\varepsilon > 0$. If $g \in C^+(X)$ with $gu \leq 1$, then there exists $\phi \in C_c^+(X)$ so that $\phi \leq g$, $g(y) - \phi(y) < \varepsilon$, and $\|f\phi\| \leq 1$ for some $f \in C(X)$ with $f \geq u$.*

Proof. Observe that we may assume $g(y) \neq 0$ and therefore can assume $\varepsilon < g(y)$. For $\eta = [2g(y)(g(y) - \varepsilon)]^{-1}\varepsilon$, Lemma 4.1 yields an $f \in C^+(X)$ with $f \geq u$ for which $f(y) - u(y) < \eta$. Now $g(y)f(y) - g(y)u(y) < \eta g(y)$ which implies $g(y)f(y) < 1 + \eta g(y)$. Define $\sigma = (\eta g(y) + 1)^{-1}g$ and note that $\sigma(y)f(y) < 1$. By Lemma 4.2, there exists $\phi \in C_c^+(X)$ with $\phi(y) = \sigma(y)$, $\phi \leq \sigma$, and $\|f\phi\| \leq 1$. Since $\sigma \leq g$, all that remains to be shown is that $g(y) - \phi(y) < \varepsilon$, but $g(y) - \phi(y) = (2g(y) - \varepsilon)^{-1}g(y)\varepsilon < \varepsilon$.

4.4. LEMMA. Assume X is locally compact and that V is a Nachbin family on X with $V \subseteq C^+(X)$. For each $v \in V$ we have $V_v^0 \subseteq v \cdot M_b(X)$, and $v \cdot \mu \in V_v^0$ if and only if $v \cdot |\mu| \in V_v^0$ whenever $\mu \in M_b(X)$.

Proof. It follows from Theorem 3.1 (and its proof) that $V_v^0 \subseteq v \cdot M_b(X)$, while it is clear that $\mu \in M_b(X)$ and $v \cdot |\mu| \in V_v^0$ implies $v \cdot \mu \in V_v^0$. Now assume $v \cdot \mu \in V_v^0$ where $\mu \in M_b(X)$ and let $U = C^+(X)$. From a result in [13], $CU_0(X) \subseteq CV_0(X)$ and the identity map $i: CU_0(X) \rightarrow CV_0(X)$ is continuous. Choose $f \in U$ with $f \geq v$ whence $i(V_f) \subseteq V_v$, and this implies $V_v^0 \subseteq (i(V_f))^0 = i^{*-1}(V_f^0)$ (i^* is the transpose of i); i.e., $i^*(v \cdot \mu) \in V_f^0$. It follows from [13, Lemma 3.8] that $V_f^0 = f \cdot B$ where $B = \{\sigma \in M_b(X) : \|\sigma\| \leq 1\}$, and hence there is a $\sigma \in B$ so that $i^*(v \cdot \mu) = f \cdot \sigma$. But $\sigma \in B$ implies $|\sigma| \in B$, and this implies $f \cdot |\sigma| \in V_f^0$. Now there is a μ -measurable function h on X with $|h| = 1$ and $\mu = h \cdot |\mu|$ [11, p. 126], and hence $v \cdot \mu = i^*(v \cdot \mu) = (vh) \cdot |\mu| = f \cdot \sigma = (fh') \cdot |\sigma|$ (where h' is the counterpart of h for σ). Consequently,

$$f \cdot |\sigma| = (vhh') \cdot |\mu| \quad \text{which implies } vhh' \geq 0 \text{ a.e. } (|\mu|),$$

and thus

$$v = |v| = |vhh'| = vhh' \text{ a.e. } (|\mu|).$$

Therefore $v \cdot |\mu| = f \cdot |\sigma|$ and we have that $v \cdot |\mu| \in V_f^0$ for every $f \in U$ with $f \geq v$. If $g \in V_v$ with $g \geq 0$, then for each $x \in X$ we have from Lemma 4.3 that $g(x) = \sup \{\phi(x) : \phi \in A\}$ where

$$A = \{\phi \in C_c^+(X) : \phi \leq g, \phi \in V_f \text{ for some } f \in U \text{ with } f \geq v\},$$

and since A is directed upward, $\int gvd|\mu| = \sup \{\int \phi vd|\mu| : \phi \in A\}$ [5, p. 184]. But $\phi \in A$ implies $\int \phi vd|\mu| \leq 1$ since $v \cdot |\mu| \in \bigcap \{V_f^0 : f \in U, f \geq v\}$, and hence $\int gvd|\mu| \leq 1$. Clearly, this gives us that $v \cdot |\mu| \in V_v^0$ which completes the proof.

4.5. THEOREM. If X is locally compact and if V is a Nachbin family on X with $V \subseteq C^+(X)$, then $V_v^0 = v \cdot B$ for each $v \in V$ where $B = \{\mu \in M_b(X) : \|\mu\| \leq 1\}$.

Proof. By Lemma 4.4, $V_v^0 \subseteq v \cdot M_b(X)$ and hence if $\nu \in V_v^0$, then there is a $\mu \in M_b(X)$ such that $\nu = v \cdot \mu$. Lemma 4.4 now gives us that $v \cdot |\mu| \in V_v^0$, and this fact together with Lemma 3.3 yield the conclusion of the theorem.

We now proceed to characterize the extremal points $\mathcal{E}(V_v^0)$ of the sets V_v^0 whenever X and V satisfy the hypotheses of Theorem 4.5. In the sequel, the point mass at $x \in X$ will be denoted by $\delta(x)$.

4.6. THEOREM. If X is locally compact and if V is a Nachbin family on X with $V \subseteq C^+(X)$, then

$$\mathcal{E}(V_v^0) = \{\lambda v(x)\delta(x) : x \in N(v), |\lambda| = 1\}$$

for each $v \in V$.

Proof. Fix $v \in V$, let $E = \{\lambda v(x)\delta(x) : x \in N(v), |\lambda| = 1\}$, and note that $E \subseteq V_v^0$. We first show that $E \subseteq \mathcal{E}(V_v^0)$, and to this end choose $\mu = \lambda v(x)\delta(x) \in E$ and suppose

$\mu = (1/2)(\sigma + \tau)$ where $\sigma, \tau \in V_v^0$. Now $\sigma = v \cdot \alpha, \tau = v \cdot \beta$ where $\alpha, \beta \in M_b(X)$ with $\|\alpha\|, \|\beta\| \leq 1$ by Theorem 4.5, and without loss of generality we may assume $|\alpha|(X \setminus N(v)) = |\beta|(X \setminus N(v)) = 0$. Thus $\lambda v(x)\delta(x) = ((1/2)v) \cdot (\alpha + \beta)$, and since it will suffice to assume $\lambda = 1$, we claim that in this case $\delta(x) = (1/2)(\alpha + \beta)$. To see this, let A be open with $x \notin A$ and choose $\phi \in C_c^+(X)$ with $\text{spt}(\phi) \subseteq A$. Then

$$\begin{aligned} \int \phi d(\alpha + \beta) &= \int_{N(v)} \phi d(\alpha + \beta) = \int_{N(v)} (\phi/v) d(\sigma + \tau) \\ &= 2v(x) \int_{N(v)} (\phi/v) d\delta(x) = 2\phi(x) = 0, \end{aligned}$$

which implies $|\alpha + \beta|(A) = 0$ [5, p. 202]. Hence $\text{spt}(\alpha + \beta) = \{x\}$, while $(\alpha + \beta)(\{x\}) = v(x)^{-1}[v(x)(\alpha + \beta)(\{x\})] = 2\delta(x)$. Since $\delta(x)$ is an extremal point of the unit ball of $M_b(X)$, $\delta(x) = \alpha = \beta$ from whence $v \cdot \delta(x) = v \cdot \alpha = v \cdot \beta$, and this implies $\mu \in \mathcal{E}(V_v^0)$.

We used above a characterization of the extremal points of the unit ball in $M_b(X)$ [3] which is an extension of the well-known Arens-Kelley result. To complete the proof, we adapt the argument which Conway [3] used to obtain this extension. Now take $\mu \in \mathcal{E}(V_v^0)$ and, using Theorem 4.5, write $\mu = v \cdot \nu$ where $\nu \in M_b(X)$ with $\|\nu\| \leq 1$ and $|\nu|(X \setminus N(v)) = 0$. Note that in this case $\|\nu\| = 1$. Consider $\mathcal{K} = \{K \subseteq X : K \text{ is compact, } |\nu|(K) > 0\}$ and observe that \mathcal{K} is not void. If $K_1, K_2 \in \mathcal{K}$ and if $K_3 = X \setminus (K_1 \cup K_2)$, then $\mu = \sum_{i=1}^3 |\nu|(K_i) v \cdot \mu_i$ where

$$\begin{aligned} \mu_i(A) &= |\nu|(K_i)^{-1} \nu(A \cap K_i), & |\nu|(K_i) &\neq 0, \\ &= 0, & |\nu|(K_i) &= 0, \end{aligned} \quad i = 1, 2, 3,$$

for Borel sets A . Hence $\mu = v \cdot \mu_1$ which implies $|\nu|(K_2) = |\mu_1|(K_2 \cap N(v)) = |\nu|(K_1)^{-1} |\nu|(K_1 \cap K_2)$ from whence $K_1 \cap K_2$ is not void. If $K \subseteq K_1 \setminus K_2$ is compact, then $|\nu|(K) = 0$, and this implies $|\nu|(K_1 \setminus K_2) = 0$; therefore $|\nu|(K_1 \cap K_2) = |\nu|(K_1) > 0$, and thus $K_1 \cap K_2 \in \mathcal{K}$. It easily follows that \mathcal{K} is closed under finite intersections and hence there is an $x \in \bigcap \{K : K \in \mathcal{K}\}$. If A is open and $x \notin A$ and if $\phi \in C_c^+(X)$ with $\text{spt}(\phi) \subseteq A$, then $|\int \phi d\mu| \leq \|\phi v\| |\nu|(\text{spt}(\phi)) = 0$, and from this we have that $\text{spt}(\mu) = \{x\}$. Now $\mu \neq 0$ implies $x \in N(v)$, and so $\mu = \alpha \delta(x)$ where $\alpha = v(x)\nu(X)$, and $|\nu(X)| = 1$ since $\|\nu\| = 1$. Consequently, $\mu \in E$ and the proof is complete.

5. Applications and observations. One of the more immediate applications of the preceding development is the following generalization (see Example 3.4) of the author's representation theorem for tensor products of weighted spaces [13]. The proof will be omitted since it is exactly as in [13] with the exception of using Theorem 4.6 in place of the corresponding result in [13].

5.1. THEOREM. *Let X and Y be locally compact. If U is a Nachbin family on X with $C_c^+(X) \leq U \leq C^+(X)$ and if V is a Nachbin family on Y with $C_c^+(Y) \leq V \leq C^+(Y)$, then the biequicontinuous completed tensor product $CU_0(X) \tilde{\otimes} CV_0(Y)$ is topologically isomorphic to $CW_0(X \times Y)$ where $W = \{u \times v : u \in U, v \in V\}$ (here $u \times v$ is the function on $X \times Y$ defined by $u \times v(x, y) = u(x)v(y)$).*

We now consider a class of Nachbin families V on a locally compact space X which have the property that $V \cdot M_b(X) = M_b(X)$. In particular, if $V \subseteq B^+(X)$, then surely $V \cdot M_b(X) \subseteq M_b(X)$, while the fact that $M_b(X) = C_0^+(X) \cdot M_b(X)$ [1, p. 99] implies every $\mu \in M_b(X)$ defines a continuous linear functional on $CV_0(X)$ whenever $C_0^+(X) \leq V$ in view of [13, Theorem 3.1]. An application of Theorem 3.1 now yields the following result.

5.2. THEOREM. *If X is locally compact and if V is a Nachbin family on X with $C_0^+(X) \leq V \subseteq B^+(X)$, then $CV_0(X)^* = V \cdot M_b(X) = M_b(X)$.*

This result leads us to the question posed below. Buck [1] asked if $(C_b(X), \beta)$ is a Mackey space, and Conway [4] gave an affirmative answer in case X is paracompact. So if X is locally compact and paracompact, then $\tau(C_b(X), M_b(X))$ is the weighted topology β determined by the Nachbin family $C_0^+(X)$. This raises the question of when $\tau(C_b(X), M_b(X))$ is a weighted topology determined by a Nachbin family V on X for which $CV_0(X) = C_b(X)$. The answer to this question (see Theorem 5.5) is an immediate consequence of the following two lemmas.

5.3. LEMMA. *If X is locally compact and if V is the set of all nonnegative u.s.c. functions on X which vanish at infinity, then V is a Nachbin family on X and $V \approx C_0^+(X)$.*

We remark before beginning the proof that it is of some interest to consider this result in light of Example 3.5.

Proof. It is, of course, clear that V is a Nachbin family on X and that $C_0^+(X) \leq V$. If $v \in V$, then $v \in B(X)$, and so to complete the proof it will suffice to assume $\|v\| = 1$ and show there is a $\phi \in C_0^+(X)$ such that $v \leq \phi$. Now $K_n = \{x \in X : v(x) \geq 2^{-n}\}$ is compact for each $n = 1, 2, \dots$, and hence for each n there is a $\phi_n \in C_c(X)$ such that $0 \leq \phi_n \leq 1$ and $\phi_n(K_n) = 1$. If we define $\phi = \sum_{n=1}^{\infty} 2^{-(n-1)} \phi_n$, then $\phi \in C_0^+(X)$. Moreover, if $x \in N(v)$, then there exists an n such that $x \in K_n$ and $v(x) \leq 2^{-(n-1)}$, while in this case $\phi(x) \geq 2^{-(n-1)}$.

5.4. LEMMA. *Let X be locally compact and let V be a Nachbin family on X . If $CV_0(X) = C_b(X)$ and if ω_V is at least as fine as β , then $V \approx C_0^+(X)$.*

Proof. It follows from [13, Theorem 3.3] that $C_0^+(X) \leq V$, while $V \leq C_0^+(X)$ follows from Lemma 5.3 together with the fact that $1 \in CV_0(X)$.

5.5. THEOREM. *If X is locally compact, then the following are equivalent:*

- (1) $(C_b(X), \beta)$ is a Mackey space.
- (2) There exists a Nachbin family V on X such that

$$CV_0(X) = (C_b(X), \tau(C_b(X), M_b(X))).$$

Several other questions remain unanswered, however. For example, one might ask which pairs (V, X) where V is a Nachbin family on X imply $CV_0(X)$ is a Mackey space, a barrelled space, etc.

Theorem 5.2 gave a class of Nachbin families V on a locally compact space X for which $CV_0(X)^* = M_b(X)$. We also note the following easily proved result.

5.6. THEOREM. *If X is locally compact and if V is a Nachbin family on X with $C_c^+(X) \leq V \leq B^+(X)$, then $M_c(X) \subseteq V \cdot M_b(X) = CV_0(X)^* \subseteq M_b(X)$ where $M_c(X)$ is the space of Radon measures with compact support. Moreover, if $V \approx C_c^+(X)$, then $CV_0(X)^* = M_c(X)$.*

Finally, consider $C(X)$ with the compact-open (co-op) topology and define $\delta: X \rightarrow (C(X), \text{co-op})^* = M_c(X)$ by $\delta(x)(f) = f(x)$. It is well known that δ is a homeomorphism onto its image when the range has its weak-* topology (e.g., see [16, p. 266]). Varadarajan [15] has obtained other results for the map δ in a somewhat different setting. In particular, he has shown that if X is a metric space, then $\delta(X)$ is sequentially closed in $((C_b(X), \|\cdot\|)^*, \sigma((C_b(X), \|\cdot\|)^*, C_b(X)))$ [15, p. 197]. With only the assumption that X be locally compact and Hausdorff, we obtain the following stronger result.

5.7. THEOREM. *Let X be locally compact and let V be a Nachbin family on X with $C_c^+(X) \leq V \leq C_0^+(X)$. If $\delta: X \rightarrow M_b(X)$ is defined by $\delta(x)$ equals the point mass at x for each $x \in X$, then δ is a one-to-one continuous closed map into $(CV_0(X)^*, \sigma(CV_0(X)^*, CV_0(X)))$.*

Proof. That $\delta(X) \subseteq CV_0(X)^*$ follows from Theorem 5.6, while it is clear that δ is one-to-one and continuous. Let F be a closed subset of X , and observe that it will suffice to show $\delta(F)$ is $\sigma(M_b(X), C_b(X))$ -closed since $CV_0(X)^* \subseteq M_b(X)$ by Theorem 5.6 while $C_b(X) \subseteq CV_0(X)$. To this end, let μ be a $\sigma(M_b(X), C_b(X))$ limit point of $\delta(F)$ in $M_b(X)$, and note that $\text{spt}(\mu)$ is not void since $1 \in C_b(X)$ implies $\mu \neq 0$. Choose $y \in \text{spt}(\mu)$ and a net $\{\delta(x_i)\}$ in $\delta(F)$ with $\delta(x_i) \rightarrow \mu$, and suppose the net $\{x_i\}$ does not converge to y . In this case there is an open neighborhood A of y for which $\{x_i\}$ is not eventually in A , and there is a $\phi \in C_c^+(X)$ such that $\text{spt}(\phi) \subseteq A$ with $|\int \phi d\mu| > 0$ [5, p. 202]. Now let $\varepsilon > 0$, choose i_0 so that $i \geq i_0$ implies $|\int \phi d\mu - \phi(x_i)| < \varepsilon$, and choose $j \geq i_0$ such that $x_j \notin A$. Thus

$$\left| \int \phi d\mu \right| \leq \left| \int \phi d\mu - \phi(x_j) \right| + \phi(x_j) < \varepsilon,$$

which contradicts the choice of ϕ . Hence $\mu = \delta(y)$ and the proof is complete.

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